THEORY OF THE ROOT LINES OF THE GENERAL POLYNOMIAL

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Short description.

This paper describes the main characteristics of the root lines of polynomials with complex coefficients. Root lines are defined as the geometric loci of the roots of the real, respective imaginary part of the polynomial. A graphical representation of these lines offers a clear picture of the positions of the polynomial's roots. The coordinates of various points of the root lines can be obtained by use of a computer program. Each root line leads to a given root, this way the polynomial can be solved and graphically represented.
A short description of the rules of variation and properties of these root lines follows depending on the position of the roots of the polynomial. This study contains many numerical examples which show not only how the root lines, but also the polynomials with complex roots and coefficients work, how they can be studied, transformed and solved.

A. Introduction

This study is the continuation and completion of a former study about the root lines presented at the 27-th ARA Congress in Oradea, Romania, on May 29 – June 2, 2002.

Length of the papers was then restricted to 5 pages, therefore some observations in that study could not be mathematically proved or demonstrated.

This paper presents all those observations and new explanations regarding the properties of root lines.

B. Importance of the root lines.

Root lines are characteristic for polynomials with complex roots. A polynomial of degree *n* has *n* roots and all roots can be complex. According to some scientific works the concept of complex roots has a particular importance to the physical and engineering sciences.
See Reference [4] Volume II, for the list of such cases.
The same [4], on page 413 and ff. also presents some curves which are root lines, but are called by the author μ(x,y) = c1 and v(x,y) = c2. (μ and v are obviously the real and imaginary part of the polynomial).

The name "root lines" of this theory is given by me, and probably is not found in any other book or article.
In the same chapter of that book [4] is also mentioned that the angles between these curves at a multiple root are equal.
It explains this with the vector dot product of the gradients.
Present study gives a more simple explanation for this.

C. The basic form of a polynomial is:

$P_n(w) = C_n w^n + C_{n-1} w^{n-1} + + C_1\ x + C_0$
(1)

where w is the independent variable of the polynomial which can be a real or a complex number; regarding w see also Par. H.
The coefficients C_n to C_0 are complex or real numbers.

D. Roots of a polynomial.
Real roots.

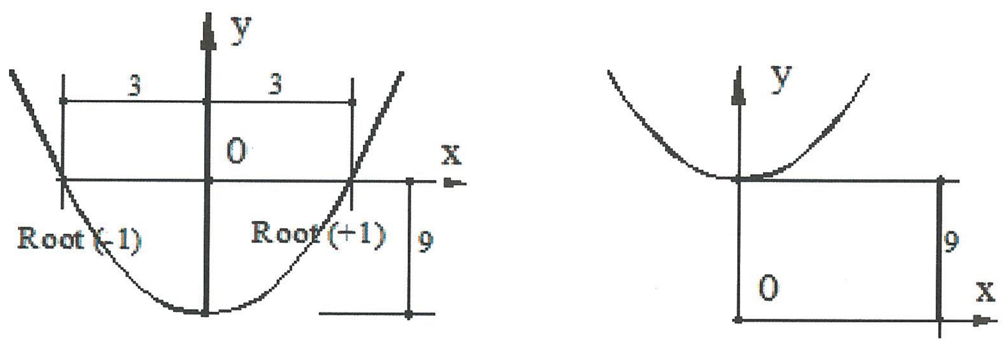
Roots (or zeros) of a polynomial are those values of the variable w for which the polynomial's value (both the real and the imaginary part) reduces to zero.
If *w* is a real number and P(w) is represented along a straight reference line, then the roots are points where the curve crosses this line.

E. Complex roots of a polynomial.

If we compare polynomial $x^2–9$ with x^2+9 then the condition $P = x^2 – 9 = 0$ can be written also $x^2 = 9$ and hence $x = +3$ or $x = –3$, so this polynomial has two real roots where the curve intersects the Ox line.

The polynomial $P = x^2 + 9$ on the other hand, as we can see from Fig. 1 has the minimum value for $x = 0$ but it never reduces to zero. See Fig. 1. From the condition $x^2+9 = 0$ results $x^2 = –9$ or $x = 3\sqrt{-1}$. But we cannot extract the square root from $–1$!

So we reach to the notion of complex numbers which have the form $a + bi$ where a and b



Polynomial $x^2 - 1$ Polynomial $x^2 + 1$

Fig. 1

are real numbers and $i = \sqrt{-1}$ is the imaginary unit.

Then the number of roots of a polynomial is completely solved: Any polynomial has n roots which are either real or complex. Actually the real roots can be considered as complex numbers with $b = 0$.

F. The second basic form of a polynomial

If all roots of a polynomial are known, then the polynomial can be written in another basic form

$$Pn(w) = (w - w_1)(w - w_2)\ldots(w - w_n) \qquad (2)$$

Where w_1, w_2, w_n are the (complex) roots and the polynomial in form (2) obviously reduces to zero if w is equal with one of the roots.

If the mathematical operations are performed, then (2) reduces to (1), because all operations are unique.

G. Independence of the roots and root lines from the reference axes.

This rule is very important for the study of the root lines.
In relation (2), if we choose any point $M(x_p, y_p)$ and calculate the value of the polynomial for this point M, then we chooses other two reference axes (Ox, Oy), the value of the polynomial in *the same point M* will be the same.
Proof: If we use Form (2) for the polynomial, then each factor is a *difference* of two distances (x_p and w_i) and both of which is measured from the same axis, so the difference remains the same.

This rule can be automatically extended to the root lines. A root line constructed in one system (of Ox, Oy) axes remains the same if the axes are moved because each point of the root line is a root. On the other hand coefficients of a polynomial change if the reference axes are moved.

H. Representation of the roots and root lines of a polynomial.

The real roots of a polynomial are represented along the Ox axis. The imaginary component bi of a complex root is assumed to be perpendicular to the real axis. So a complex number a + bi is a vector whose components define the complex variable w = a + bi = x + yi then defines a Complex Plane (See Fig 2)'

Observation.

Fig. 2 has a tri-dimensional representation. The complex variable having two components (parts). x and y, it is natural to assume that the function P(x+yi) = u+vi has also two components u (real part) and v (imaginary part) which are in general *not equal* with x or y, so it is logical to represent them in a third direction, perpendicular to the Oxy Plane, i.e. in direction z. Therefore it is better to call the complex variable x + vi w and not z.
be represented as two surfaces, each point of which has a value in direction z.

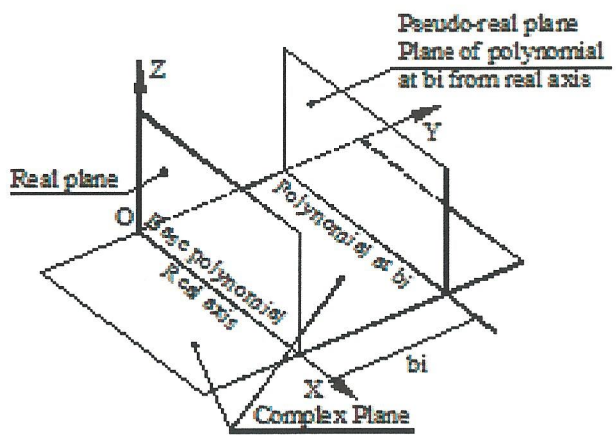
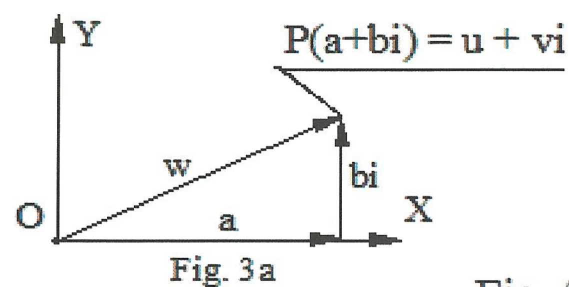


Fig. 2

The two parts of the function P(w) will be assumed then to have direction z.

I. Points of root lines can be obtained by substitution of the complex variable in the polynomial.

If we substitute the complex variable w = a + bi in a polynomial P_n, then in the same point of the Complex Plane we will obtain two values: u an vi. These values *u* and *v* are assumed to be vertical, i.e. perpendicular to the complex plane Oxy in the direction of the Oz axis (See Fig. 3)
The imaginary unit i only very rarely disappears. Therefore *all terms multiplied by i* added up give an imaginary part noted generally with *v*. The other terms (without i) give the real part noted *u*.
If we substitute w = a + bi = x + yi in the polynomial and give b a numerical value, but leave x variable, then obviously remain just the terms with x unknown. These have the powers 0 to n. All these terms form than *two polynomials in x, one multiplied by i (Imaginary Part), and another without i (Real Part). They* correspond to *the line at distance b* (b can be chosen numerically) one being the real, the other the imaginary part). If these polynomials are solved, they give *n* points for the root lines of the real part and n for the imaginary part. (See Fig. 3b)
Only the real roots have to be calculated, the complex roots are discarded *being outside of the line at an imaginary) distance.*



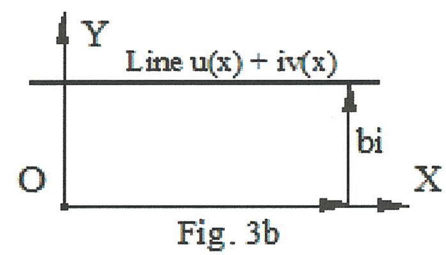


Fig. 3

J. Different possible positions of the Polynomial Center.

The polynomial centre was defined in the first part of this study (in 2002) as the geometrical centre of all roots both in direction x and y. (See [7] Par. (2), Pg. 841)
In case of a single multiple root (i.e. there are no other roots than the multiple root) all root lines are straight lines which intersect where the multiple root is. There then both the real and imaginary part reduce to zero (so this is a complex root) and the polynomial center is also there (all roots are concentrated in one point).
So in this case all root lines pass *exactly* through the polynomial center and they are *straight lines* and coincide with the asymptotes.
In case of circular (binary) polynomials in the form $w^n - 1$ the polynomial centre coincides with the center of the circle on which the roots are, this being the *best approximation* for the action of *all roots*.
More about such polynomials see [7], Par 5A2., Pg 843.
For other polynomials the asymptotes correspond to w^n because the other terms for high values of w are negligible against w^n and therefore this case, especially for points far from the center – the Polynomial Center is similar to (i.e. the best approximation for) the case of a multiple root. It is also an *invariant* of the polynomial and *depends on all roots*.
All practical cases which I studied confirmed this hypothesis.
There is however also a rigorous mathematical proof, but it is rather long to be included in a so short study.

K. Solving polynomials.

To solve polynomials even with complex roots is simple if there is an adequate method and computer program. Such a method is Newton's method in the *complex domain.*
In order to see how this method works, let's see first Newton's method in the *real domain.*
According to the real domain method, for a given x value, the polynomial's value z = P(x)

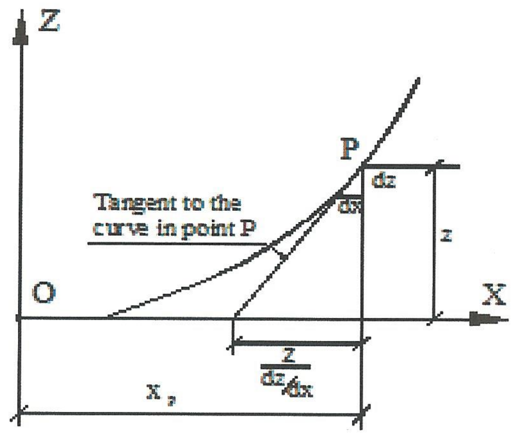


Fig. 4

and the value of its derivative is $z' = \dfrac{dP(x)}{dx} = \dfrac{dz}{dx}$ are known (see Fig. 4).
But z' is the trigonometric tangent of the geometrical tangent to the polynomial's curve. Then the point where the geometrical tangent crosses the Ox axis will be at $x - \dfrac{z}{z'}$. This is now an approximation closer to a root of the polynomial. In many manuals $\dfrac{z}{z'}$ is noted with $\dfrac{f}{f'}$ (function f and its derivative f').

K. Solving polynomials (Part II: In the complex domain)

In the complex domain, the function with a complex variable is

$f = u + vi$ and its derivative is
$f' = u' + v'i$ where f', u' and v' are derivatives of f, both with respect to x.

If we divide f with f' according to the rule of division of complex numbers we obtain

$$\frac{f}{f'} = \frac{u+vi}{u'+v'i} = \frac{uu'+vv'}{u'^2+v'_2} + \frac{vu'-uv'}{u'^2+v'^2}i$$

The obtained two fractions subtracted from an initial value of an arbitrary complex number $w = x + yi$ give a new approximation of w. This is then repeated until value of $\frac{f}{f'}$ becomes negligible.

J.B. Moore in [1] doesn't mention the name of Newton. He calls the above approximations, i.e. the value of $\frac{f}{f'}$ 'steepest descent vector' (s d v) but arrives to the same expression as given above.

For the s.d.v. see also [7] Par. 6A, Pg. 845.

The above relation inserted in a computer program gives after a few steps value of one root. Then a synthetic division has to be performed, that means the polynomial is divided by $w - w_1$ where w_1 is the value of the calculated root. After this the polynomial's degree reduces by 1.

When $n = 1$, the root's value is calculated directly.

K. Observations regarding this method.

1.) The program works well also for polynomials with real coefficients or real roots, but then the imaginary part of the coefficients has to be set to zero.

2.) Whereas Newton's method in the real domain works only if the initial guess is close to a root, there is no such restriction in case of the program in complex domain.

3.) If the program enters in an 'endless loop' (it happens extraordinarily seldom), then increase initial guess, or change sign of the imaginary part (i.e. change the initial guess 1 + i to 10 + 10i, 100 + 100i, 1 – i, 10 – 10i, 100 – 100i etc)

Note.
The size of this paper doesn't permit to give a complete listing of this or other programs.
If you are interested in more details, please write to my e-mail address.

L. Rotation of the asymptotes of a multiple root multiplied by a complex constant.

A polynomial which has only one *n* times multiple root has the form.

$P_n = w^n = (x + bi)^n$ or in the trigonometric form:
$P_n = \rho^n(\cos n\varphi + i\sin n\varphi)$

If this expression is multiplied by a complex number in the form $a + bi = a(1 + ki)$ where $k = \frac{b}{a} = \tan\alpha$ $\alpha = \tan^{-1}(k)$ (using for the argument of the complex multiplier $1 + ki$ the letter α instead of φ in order to make it different of the multiple root) then results

$$(1+ ki)\, w^n = \cos n\varphi - k\sin n\varphi + [\sin n\varphi + k\cos n\varphi]\, i$$

Equating the real part and the imaginary part separately with zero, for the real part we obtain:

$$\tan n\varphi = \frac{1}{k} = \tan(90° - \alpha) \quad \text{or } n\varphi = 90° - \alpha \quad \text{or}$$

$$\varphi_r = \frac{90^0 - \alpha}{n} + k_1 \frac{\pi}{n} \qquad (3a)$$

L. (continued) Same as above, but for the Imaginary Part. The term of multiplicity.

For the Imaginary Part we have:

$$\tan n\varphi = -k = -\tan\alpha \quad \text{or} \quad n\varphi = -\alpha \quad \text{or} \quad \varphi_i = \frac{-\alpha}{n} + k_1 \frac{\pi}{n} \tag{3b}$$

Where: $k_1 = 1, 2, \ldots n-1$ (all integer numbers).

Relations (3a) and (3b) show directions of the lines along which the Real or Imaginary Part of w^n is equal with zero, i.e. their root lines.

φ_r and φ_i are measured from the reference axis Ox

The term $k1 \frac{\pi}{n}$ in these relations is called **the term of multiplicity**, because it adds to φ_r or φ_i $n-1$ more directions of the root lines.

Proof:

If φ_r or φ_i is substituted in the expression of $P_n = w^n$ then the arguments $\varphi_r + k_1 \frac{\pi}{n}$ or $\varphi_i + k_1 \frac{\pi}{n}$ become $n\varphi_r + \pi$ resp $n\varphi_i + \pi$, but if for $n\varphi_r$ or $n\varphi_i$ $P_n = 0$ then it will be zero also for $\varphi_r + \pi$ resp $\varphi_i + \pi$

Other method:

Product of two complex numbers = $Rn\varphi R\alpha[\cos(n\varphi+\alpha) + i\sin((n\varphi+\alpha)]$

Real Part is 0 for $n\varphi_r + \alpha = 90°$ or $\varphi_r = \frac{90^0 - \alpha}{n} + k_1 \frac{\pi}{n}$

Imaginary Part is 0 for $n\varphi_i + \alpha = 0$ 0r $\varphi_i = \frac{-\alpha}{n} + k_1 \frac{\pi}{n}$

M. Consequences of relation (3a) or (3b)

Consider a polynomial which has a triple root in the origin and another root at –3 – 2i. (See Fig. 5). It can be written as

$$P_4 = x^3(x + 3 + 2i)$$

In the origin $x = 0$ and the parenthesis expression

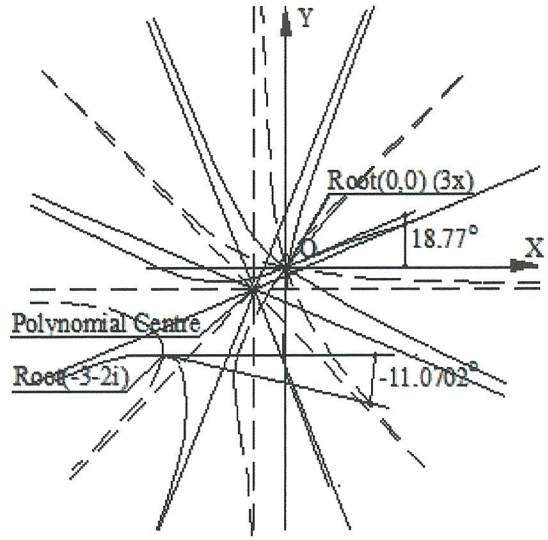


Fig. 5

reduces to the complex constant $3 + 2i = 3\left(1+\frac{2}{3}i\right)$

so in this case $k = \frac{2}{3} = .6666 \quad \alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.690067°$

This is exactly the case presented in the previous paragraph (M.), i.e. the complex multiple root x^3 ($= w^3$) is multiplied by the complex constant $3 + 2i$.

Direction of the real root line will be then given by the angle φ_r equal with (using relations from Par. L.):

$$\varphi_r = \frac{90^0 - \alpha}{n} + k_1 \frac{\pi}{3} = 18.77° + 60° \text{ where } \alpha = \tan^{-1}\left(\frac{2}{3}\right) 33.690077° \quad n = 3 \text{ (multiplicity of the directions) } k_1 = 0, 1, 2, 3$$

...

N. Observations for relations (3a) and (3b)

Note that in these relations n is the multiplicity, i.e. the number of roots in the multiple root (in this case n = 3) and *not* the order of the polynomial (which in this case is 4!)
Hence results that this polynomial has root lines which in the origin O (where in this case the triple root is) are rotated by φ_r and φ_i and to these directions are added two (in general *n-1*) more, so the total of directions in which the polynomials real or imaginary part is zero is equal with n, i.e. the order of multiplicity (in general n and in this case 3) of the root.

The factor 3 + 2i is the influence of the other root on the multiple root. It causes a rotation of *all root lines which pass through the multiple (in this case triple) root* by φ_r resp φ_i but the interval between the lines remains always the same $\left(k_1 \frac{\pi}{3}\right)$ because of the term $k_1 \frac{\pi}{3}$ in general $k_1 \frac{\pi}{n}$

Further it is easy to see that the complex multiplier is $3 + 2i = 1 + \frac{2}{3}\, i = a\left(1 + \frac{b}{a}\right)$ and

$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$ where a is the Real Part and b the Imaginary Part of the simple root.

In this concrete case (see Fig. 5) it was found that $\varphi_r = 18.77°$
This is the angle of the tangent to the first real root line with the Ox axis. The other root lines in the same point are at equal intervals of $\frac{\pi}{3}$.

O. Case of several multipliers.

In case that the polynomial has more than one complex roots in different points, these can be multiplied as complex numbers and their product is

$$\rho_1\rho_2\,\rho_3\ldots[\cos(\alpha_1+\alpha_2+\ldots)+i\,\sin(\alpha_1+\alpha_2+\ldots)]$$

The arguments of the roots can be added up to

$$\alpha_{tot}=\tan^{-1}(k_1)+\tan^{-1}(k_2)\ldots$$

Each argument is measured from the positive Ox axis.
In case of a Root Number k, best method is to set this root at the origin, but so that the relative position (i.e. distance) between the roots remains the same. (The simplest solution is then to move the reference axes and not the roots!). All roots have to enter in the sum, except the Root k.
If there is a multiple root, it is like several simple roots added up, i.e. the *argument* of the multiple root is multiplied by the number of multiplicity of the root.
The total argument α_{tot} is then used in the relations of Par. M. (instead of α).
This theory shows that the root lines at *each root* of a polynomial are rotated by a value which depends on *all other roots* of the polynomial. This has some similarity with a system of celestial bodies (like the sun and its planets) where each planet influences the others, but in case of a polynomial the influence depends only on the *angle* of the root, relative to the other root. The angles of all roots are then added up numerically.

P. Rotation of the root lines at a simple root.

With the same relation as in Par. M. to O. we can calculate also the rotation of the root lines at the simple root 3 + 2i.
The 3 times multiple root is equivalent with three simple roots. (See also observations in the previous paragraph O. regarding multiple multipliers).
For one root we have

$$\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.69^{o}$$

For three roots $\alpha_{tot} = 3 . \tan^{-1}\left(\frac{2}{3}\right) = 3 \cdot 33.69 = 101.07^{o}.$

Then $\varphi_r = \dfrac{90^{o} - \alpha_{tot}}{n} = \dfrac{90 - 101.07}{1} = -11.07^{0}$

See again Fig. 5.

Q. Polynomials multiplied by a complex number.

The first coefficient (i.e. that of x^n) is in most cases of polynomials equal with 1 and $C_{ny} = 0$ (i.e. C_n has no imaginary part). If a polynomial is given in form (2) and the mathematical operations are performed to obtain form (1), then C_n will result = 1 (real), even if all roots are complex.
So if $C_{ny} \neq 0$ then it can be assumed that the polynomial was multiplied by: $C_{nx} + i\, C_{ny}$.
This complex factor then produces a rotation of the asymptotes which can be calculated with Rel. (3) where in this case: $k = \frac{C_{ny}}{C_{nx}}$ $\alpha = \tan^{-1}(k)$.

R. Other method to eliminate rotation.

Or, another (better) method is to divide the whole polynomial by the first, complex coefficient $C_{nx} + i\, C_{ny}$. After this division $C_{nx} = 1$ $C_{ny} = 0$ and the polynomial will be a 'regular' polynomial.
$P_n = d_v\, P_d$ d_v = divisor P_d divided polynomial
The roots remain after division the same, because the polynomial is multiplied just by a constant.
The Polynomial Center can be calculated from the divided polynomial. It is the same for both polynomials, because it is a function of *all roots*, and these are the same for both.

S. Numerical example for the previous case:

As an example consider the following polynomial, given in Ref [5]

$$2x^4-30x^3+163x^2-1773+(3x^4+2x^3+472x+4208)i$$

In this case C_{ny} is not 0, $k=\frac{C_{ny}}{C_{nx}}=\frac{3}{2}=1.5$ $\alpha=\tan^{-1}(1.5)=$ therefore all asymptotes are rotated by

$$\varphi_r=\frac{90^0-\tan^{-1}(1.5)}{4}=8.422517°$$

φ_r is measured from the Ox axis.

T. Dividing the polynomial

If the whole polynomial (i.e. all coefficients) are divided by the first coefficient, i.e. $C_{x4}+C_{y4}\,i=2+3i$ then we obtain the following polynomial:

$$13x^4-54x^3+326x^2+1416x+9078+(94x^3-489x^2+944x+13735)i \quad \text{or}$$
$$x^4-4.15384815x^3+25.076923x^2+108.923077x\ 698.307692+(7.230769x^3-37.615385x^2+72.615385x+1056.5385)i.$$

U. Characteristics of the reduced polynomial

This polynomial has the asymptotes according to the general rule (first real asymptote at $\frac{\pi}{2n}$) and the same roots. Hence results the polynomial centre as:

$$PolCtr_x=-\frac{C_x(n-1)}{nCx(n)}=\frac{54}{13\cdot 4}=1.0384615 \qquad PolCtr_y=-\frac{C_y(n-1)}{nCx(n)}=-\frac{94}{13\cdot 4}=-1.807692$$

V. Sum of the angles of the roots of a polynomial related to a point M on a root line of the Real Part.

If the sum of the angles of all roots of a polynomial related to a point M is equal with 90° then that point is situated on a root line of the Real Part.

W. Sum of the angles of the roots of a polynomial related to a point M on a root line of the Imaginary Part.

If the sum of the angles of all roots of a polynomial related to a point M is equal with 90° then that point is situated on a root line of the Imaginary Part.

X. Proof of Par. V and W.

If the point M is placed in the origin then the value of the polynomial in point M is equal with

$$P = (-a_1 - b_1\, i)(-a_2 - b_2 i) \quad (-a_n - b_n\, i) \ldots$$

Or written in trigonometric form

$$P = \rho_1 \rho_2\, \rho_3 \ldots [\cos(\varphi_1 + \varphi_2 + \ldots) + i\, \sin(\varphi_1 + \varphi_2 + \ldots)]$$

From this expression it is obvious, that if the sum of the angles $\varphi_1, \varphi_2 \ldots \varphi_n$ is equal with 90° then the Real Part of the polynomial is zero and if the sum is 0 then its imaginary part is zero (in point M).

U2. Consequence of Paragraph V and W

If all roots of a polynomial are rotated by an angle φ as a rigid body around the polynomial centre, then if φ points in a principal direction then all root lines of the polynomial rotate by the angle φ but their status could be reversed (i.e. a root line which was of the real part becomes of the imaginary part and vice-versa) if the direction defined by φ corresponds to another principal direction.

V2. Proof of paragraph U2.

Because of the rigid rotation, the angle at which a root is seen remains the same as before, but the angle φ is added to it n times (for each root). This additional value of n_φ may cause that the total angle to be a multiple of 90°.
In case of Fig. 6 the rotation being 54°, $n_\varphi = 5{\cdot}54 = 270° = 180° + 90°$. The 180 ° doesn't change anything, but the 90° changes. If sum of the angles initially was 0, now it becomes 90°, and if it was 90°, now it will be 180°, that means all root lines will change their status.

W2. Example for paragraphs U2-V2.

As an example consider a polynomial with real roots at x = 1, x = 3 and x = 6 and a pair of complex roots x = 8 + .5i and 8 – .5i. This polynomial has the equation:

$$x^5 - 25x^4 + 225x^3 - 903x^2 + 1566x - 864 + (-6x4 + 106x3 - 622x2 + 1350x - 828)\,i$$

If all roots are rotated by 54° around the polynomial centre (at x = 5.2) in positive direction then the roots will be:

2.7313019 – 3.397871i 3.906872445 –1.779837386i
5.6702282 + .647213595i 6.4412902 +2.55914921i and
7.250307204 + 1.971354968 i

These roots correspond to following polynomial (using Rel. [2])

$$x^5 - 26x^4 + 276.3176748x^3 - 1572.5094075x^2 + 4222.1774568x - 4701.3827155856$$
$$+ (-18.2127322x^3 + 288.704436236x^2 - 1569.03213486x + 2973.05628351)\,i$$

W3. Characteristics of the rotated root lines

This polynomial has the same root lines as the previous one but the status of the root lines are reversed, i.e. if a root line was before real, now it will be imaginary and vice–versa.
See also the proof at the beginning of this paragraph.

Observe that the rotation of 54° is an integer multiple of $\frac{\pi}{2n} = 18$ which is the interval between two different root lines (one of the real part and one of the imaginary part)
See Fig 6. and Fig. 7. for the initial and rotated polynomial.

X.

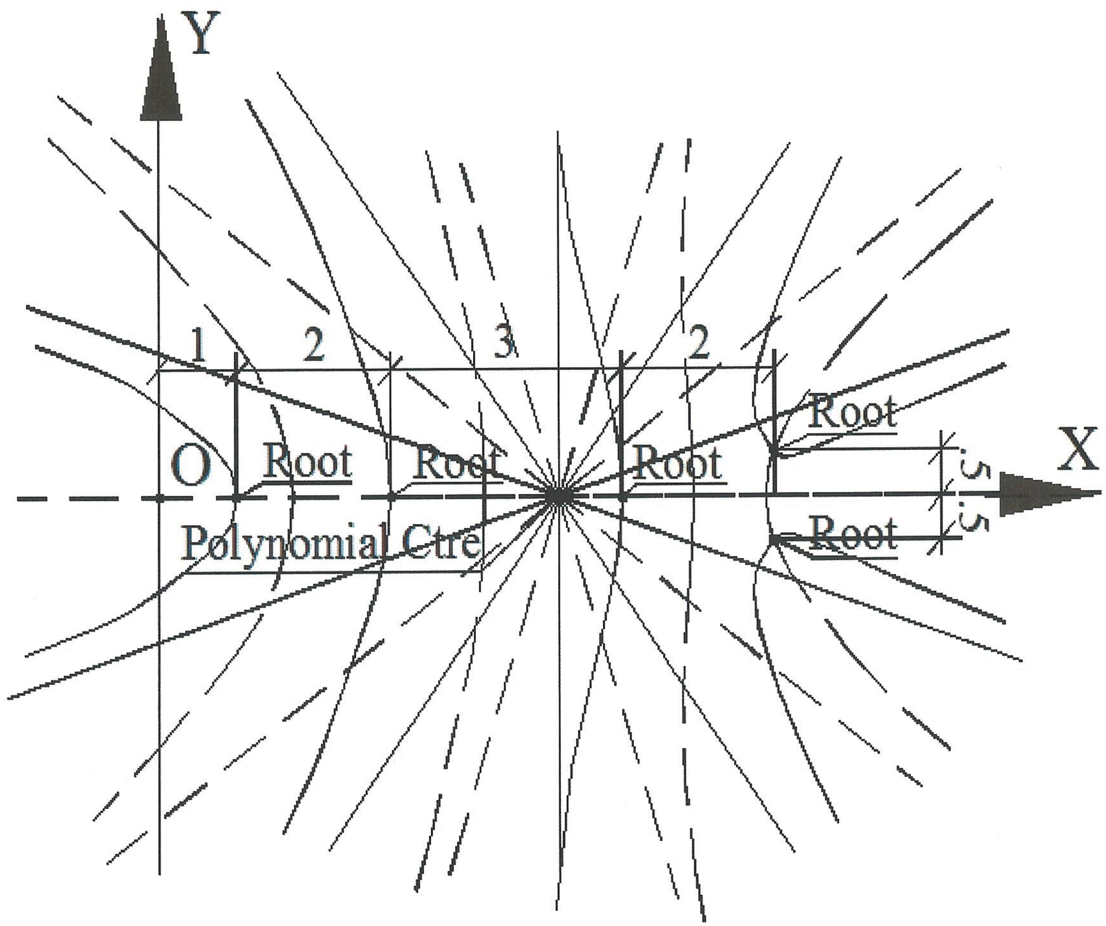
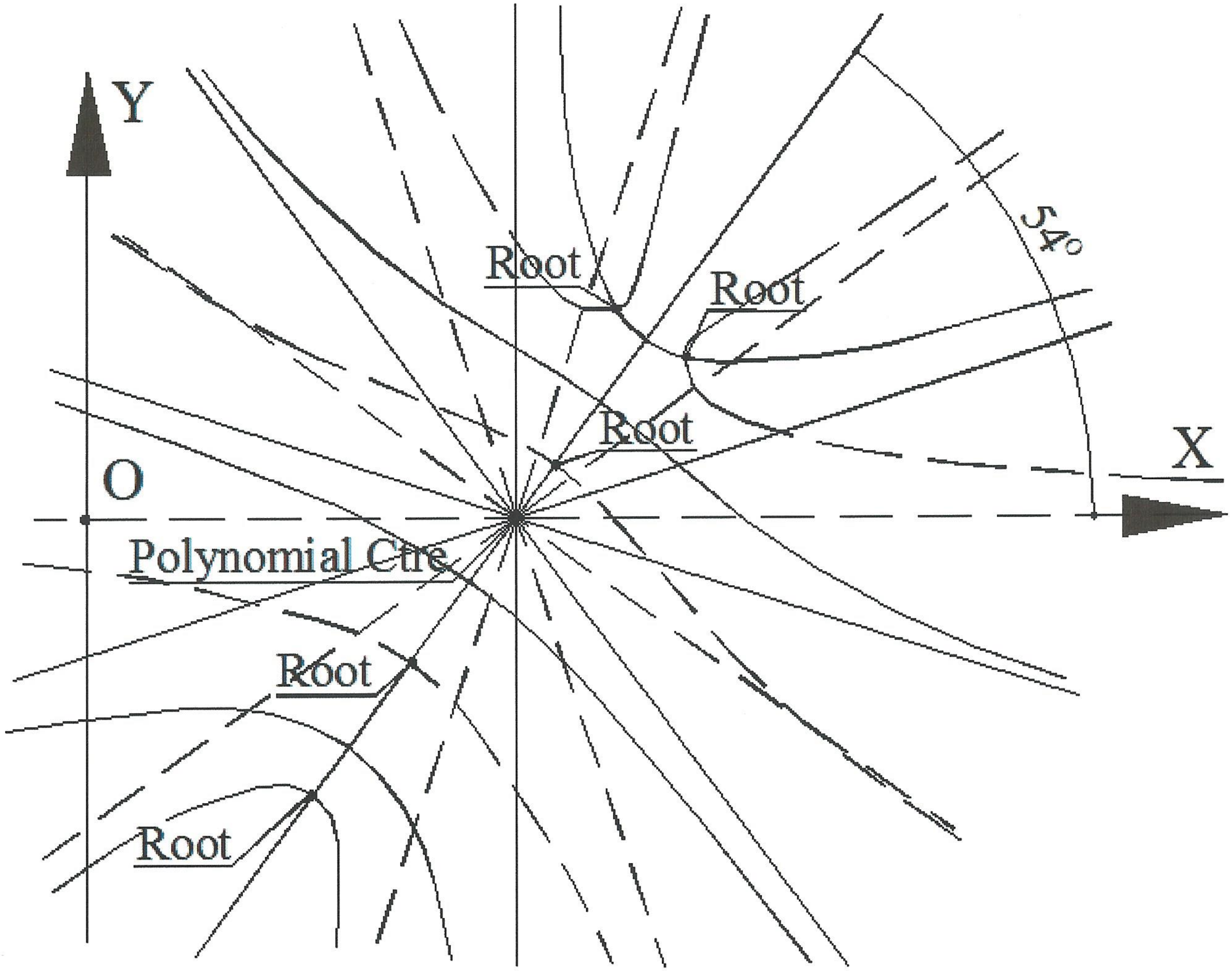


Fig. 6

a. The original roots on the Ox axis

Y.



b. The roots rotated by 54°

Fig. 7

Z. References

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Thank you